Problem 1.34

Prove that in the absence of external forces, the total *angular* momentum (defined as $\mathbf{L} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha$) of an N-particle system is conserved. [*Hints*: You need to mimic the argument from (1.25) to (1.29). In this case you need more than Newton's third law: In addition you need to assume that the interparticle forces are *central*; that is, $\mathbf{F}_{\alpha\beta}$ acts along the line joining particles α and β. A full discussion of angular momentum is given in Chapter 3.]

Solution

The solution is written in Section 3.5, which goes from page 93 to page 95. The angular momentum of a particle is defined on page 90.

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$$

Suppose there are N particles in space and define a coordinate system with origin O. Consequently, every particle has a position vector. There are interparticle forces between them all (F_{ij} is the force on particle i from particle j) and external forces acting on each of them (F_i^{ext} is the external force acting on particle i). The total angular momentum is the vector sum of all individual angular momenta.

$$\mathbf{L} = \sum_{\alpha=1}^{N} \boldsymbol{\ell}_\alpha = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{p}_\alpha$$

Take the derivative of both sides with respect to time.

$$\frac{d}{dt}(\mathbf{L}) = \frac{d}{dt} \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{p}_\alpha$$

$$\frac{d\mathbf{L}}{dt} = \sum_{\alpha=1}^{N} \frac{d}{dt}(\mathbf{r}_\alpha \times \mathbf{p}_\alpha)$$

$$= \sum_{\alpha=1}^{N} \left(\frac{d\mathbf{r}_\alpha}{dt} \times \mathbf{p}_\alpha + \mathbf{r}_\alpha \times \frac{d\mathbf{p}_\alpha}{dt} \right)$$

$$= \sum_{\alpha=1}^{N} \left(\mathbf{v}_\alpha \times \mathbf{p}_\alpha + \mathbf{r}_\alpha \times \frac{d\mathbf{p}_\alpha}{dt} \right)$$

$$= \sum_{\alpha=1}^{N} \left[\mathbf{v}_\alpha \times (m_\alpha \mathbf{v}_\alpha) + \mathbf{r}_\alpha \times \frac{d\mathbf{p}_\alpha}{dt} \right]$$

$$= \sum_{\alpha=1}^{N} \left[m_\alpha \underbrace{(\mathbf{v}_\alpha \times \mathbf{v}_\alpha)}_{=\mathbf{0}} + \mathbf{r}_\alpha \times \frac{d\mathbf{p}_\alpha}{dt} \right]$$

$$= \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \frac{d\mathbf{p}_\alpha}{dt}$$

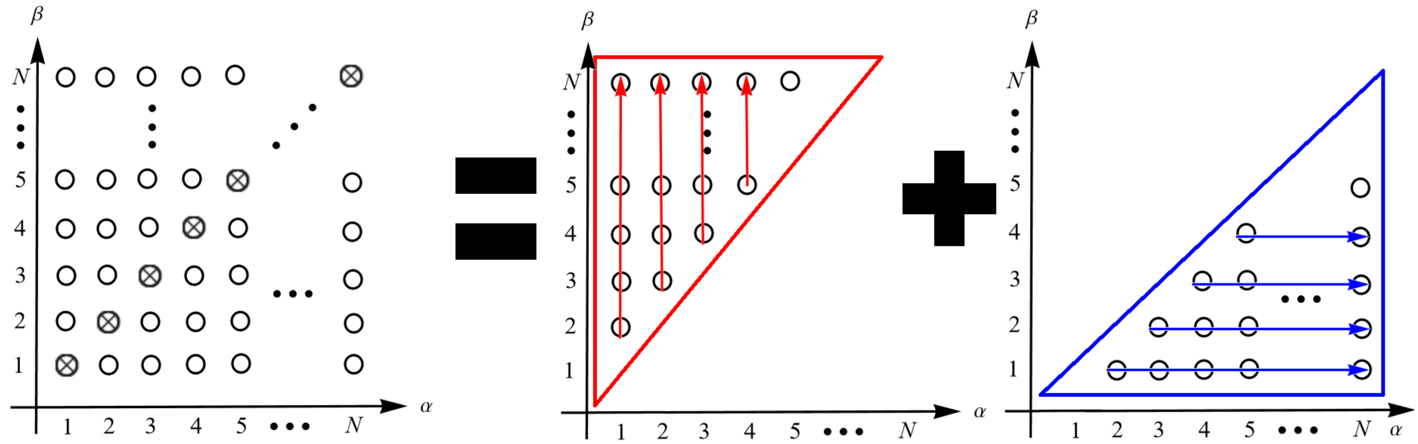
According to Newton's second law, the rate of change of momentum is equal to the net force on a particle.

$$\frac{d\mathbf{L}}{dt} = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \left(\sum \mathbf{F}\right)_\alpha$$

$$= \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \left(\sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}}\right)$$

$$= \sum_{\alpha=1}^{N} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} + \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}} \tag{1}$$

In order to simplify the double sum, visualize the points in the $\alpha\beta$-plane being summed over.



$$\sum_{\alpha=1}^{N} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} = \sum_{\alpha=1}^{N-1} \sum_{\beta=\alpha+1}^{N} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} + \sum_{\beta=1}^{N-1} \sum_{\alpha=\beta+1}^{N} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta}$$

Let α be β, and let β be α in the second double sum.

$$\sum_{\alpha=1}^{N} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} = \sum_{\alpha=1}^{N-1} \sum_{\beta=\alpha+1}^{N} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} + \sum_{\alpha=1}^{N-1} \sum_{\beta=\alpha+1}^{N} \mathbf{r}_\beta \times \mathbf{F}_{\beta\alpha}$$

$$= \sum_{\alpha=1}^{N-1} \sum_{\beta=\alpha+1}^{N} (\mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} + \mathbf{r}_\beta \times \mathbf{F}_{\beta\alpha})$$

By Newton's third law, $\mathbf{F}_{\beta\alpha} = -\mathbf{F}_{\alpha\beta}$.

$$\begin{aligned}
\sum_{\alpha=1}^{N}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} &= \sum_{\alpha=1}^{N-1}\sum_{\beta=\alpha+1}^{N} [\mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} + \mathbf{r}_\beta \times (-\mathbf{F}_{\alpha\beta})] \\
&= \sum_{\alpha=1}^{N-1}\sum_{\beta=\alpha+1}^{N} (\mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta} - \mathbf{r}_\beta \times \mathbf{F}_{\alpha\beta}) \\
&= \sum_{\alpha=1}^{N-1}\sum_{\beta=\alpha+1}^{N} \underbrace{(\mathbf{r}_\alpha - \mathbf{r}_\beta) \times \mathbf{F}_{\alpha\beta}}_{=\,\mathbf{0}} \\
&= \mathbf{0}
\end{aligned}$$

$\mathbf{r}_\alpha - \mathbf{r}_\beta$ is the displacement vector from particle β to particle α, and $\mathbf{F}_{\alpha\beta}$ is the (central) interparticle force from particle β to particle α. These have the same direction, so their cross product is the zero vector. Equation (1) then becomes

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \underbrace{\sum_{\alpha=1}^{N}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta}}_{=\,\mathbf{0}} + \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}} \qquad (1) \\
&= \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}}.
\end{aligned}$$

Therefore, in the absence of external forces,

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times (\mathbf{0}) \\
&= \mathbf{0},
\end{aligned}$$

which means that the total angular momentum of an N-particle system is conserved:

$$\mathbf{L}_{\text{initial}} = \mathbf{L}_{\text{final}} = \textbf{constant}.$$